THE SYNCHRONIZATION OF NEARLY-SIMILAR DYNAMIC SYSTEMS CLOSE TO LIAPUNOV SYSTEMS

(O SINKHRONIZATSII POCHTI ODINAKOVYKH DINAMICHESKIKH SISTEM, BLIZKIKH K SISTEMAM LIAPUNOVA)

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R.F. NAGAEV
(Leningrad)

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The theory of synchronization studies the behavior of interconnected systems of objects. In its simplest case the problem is to ascertain the conditions for the existence and the stability of periodic states. The general statement of the problem, and also numerous examples of the appearance of synchronization in nature and technology are given in the paper by Blekhman [1]. The problem of internal synchronization of a system of dynamic objects under the action of weak linear couplings was considered by us in paper [2].

It is assumed below that the motion of an isolated object is described by a system of differential equations which is close to a Liapunov system [3]. Consequently, in isolated objects for the generating approximation we realize, generally speaking, a nonisochronous periodic state whose period varies in some finite or infinite range depending upon a certain parameter associated in some way or other with the initial conditions. In order to have the possibility of adjusting the object frequency by means of that of the external periodic perturbation being transmitted to the object by a weak coupling, in this case it is required only that this frequency be included in the frequency range of the isolated object. For internal syncronization, i.e. the synchronization of the self-contained interconnected system of objects on the whole, it is required, naturally, that their frequency ranges intersect. It is apparent that for systems having such objects the tendency toward synchronization is the strongest.

The paper consists of four sections. In the first two sections the problem is considered in the general formulation for the case of nearly similar objects. Existence conditions and the necessary stability conditions are derived for the synchronous states. In the last two sections the resuts obtained are used to investigate the synchronization of self-phasing of systems of almost-conservative objects which are located on a supporting body of a sufficiently general form. A generalized integral stability criterion is established for such a system.

1. In contrast to paper [2], we shall merely write down the equations of motion of the interconnected system of objects with excluded coupling coordinates. In other words, we shall assume that the coupling coordinates are in the form of known functions of the object coordinates. However, for the case when resonance in the coupling coordinates is absent, this assumption is quite rigorous.

Thus, we shall assume that the motion of the interconnected objects is described by the system of equations

$$\frac{dx_{si}}{dt} = X_s(x_i) + \mu f_{si}(x_1, \ldots, x_m, t, \mu) \qquad \begin{pmatrix} s = 1, \ldots, n \\ i = 1, \ldots, m \end{pmatrix} \tag{1.1}$$

Here m is the total number of objects, while n is the order of the system describing the motion of each object. In system (1.1) for brevity we denote

$$X_s(x_i) = X_s(x_{1i}, \ldots, x_{ni})$$

$$f_{si}(x_1, \ldots, x_m, t, \mu) = f_{si}(x_{11}, \ldots; x_{n1}; \ldots; x_{1m}, \ldots, x_{nm}; t, \mu)$$

When $\mu = 0$ we have m unconnected similar Liapunov systems

$$\frac{d\overset{\circ}{x}_{si}}{dt} = X_s(x_i^\circ) \qquad \begin{pmatrix} s = 1, \ldots, n \\ j = 1, \ldots, m \end{pmatrix} \tag{1.2}$$

admitting of a family of periodic solutions

$$\overset{\circ}{x}_{si} = \varphi_s(t + \alpha_i, c) \tag{1.3}$$

of period $T = T(c)$, depending on $m+1$ arbitrary real constants $\alpha_1, \ldots, \alpha_m, c$, and defined in a certain region of the phase space of the system.

The coupling functions f_{si} are assumed to be continuous in the variables $x_1, \ldots, x_m$ on all the trajectories of the generating solution (1.3), their period is 2π in the time variable t, and they are analytic with respect to the positive parameter μ, if μ is less than some μ_0.

The variational equations of the generating system (1.2),

$$\frac{d\xi_{si}}{dt} = p_{s1}(t + \alpha_i)\,\xi_{1i} + \ldots + p_{sn}(t + \alpha_i)\,\xi_{ni} \qquad \begin{pmatrix} s = 1, \ldots, n \\ i = 1, \ldots, m \end{pmatrix} \tag{1.4}$$

where

$$p_{sr}(t) = \frac{\partial X_s[\varphi(t, c)]}{\partial \varphi_r(t, c)} \tag{1.5}$$

admit of m independent $T(c)$-periodic solutions

$$\xi_{si}^{(l)} = \varphi_s^{\cdot}(t + \alpha_i, c)\,\delta_{il} \qquad (l = 1, \ldots, m) \tag{1.6}$$

which are obtained by differentiation of (1.3) with respect of the phases $\alpha_1, \ldots, \alpha_m$ in sequence, and admit of m linearly growing solutions

$$\xi_{si}^{(l)} = \left[-\frac{t + \alpha_i}{T}\frac{dT(c)}{dc}\varphi_s^{\cdot}(t + \alpha_i, c) + y_s(t + \alpha_i, c)\right]\delta_{il} \qquad (l = 1, \ldots, m) \tag{1.7}$$

The latter solution can be obtained if we take the derivative with respect to c of the generating solution (1.3) as the solution of one of the subsystems of (1.4), while the solutions of the other subsystems are set equal to zero. The $T(c)$-periodic functions $y_s(t,c)$ have the form

$$y_s(t, c) = \frac{\partial}{\partial c'}\varphi_s\left[\frac{T(c')}{T(c)}t, c'\right]\Bigg|_{c'=c} \tag{1.8}$$

and will be the solution of the inhomogeneous system

$$\frac{dy_s}{dt} = p_{s1}y_1 + \ldots + p_{sn}y_n + \frac{1}{T(c)}\frac{dT(c)}{dc}\varphi_s^{\cdot}(t, c) \qquad (s = 1, \ldots, n) \quad (1.9)$$

under the initial conditions

$$y_s(0, c) = \frac{\partial \varphi_s(0, c)}{\partial c} \quad (1.10)$$

The system of equations adjoint to (1.4)

$$\frac{d\zeta_{si}}{dt} + p_{1s}(t + \alpha_i)\zeta_{1i} + \ldots + p_{ns}(t + \alpha_i)\zeta_{mi} = 0 \begin{pmatrix} s = 1, \ldots, n \\ i = 1, \ldots, m \end{pmatrix} \quad (1.11)$$

also has m periodic solutions with period $T(c)$,

$$\zeta_{si}^{(l)} = \psi_s(t + \alpha_i, c)\,\delta_{il} \qquad (l = 1, \ldots, m) \quad (1.12)$$

which, as a consequence of the existence of $T(c)$-periodic solution of system (1.9), satisfy condition

$$\sum_{s=1}^{n} \varphi_s^{\cdot}(t, c)\,\psi_s(t, c) = 0 \quad (1.13)$$

If c is a simple root of Equation

$$T(c) = 2\pi \quad (1.14)$$

then the condition for the existence of a synchronous state in the interconnected system of objects [3] is written in the form

$$P_l(\alpha_1, \ldots, \alpha_m) = \sum_{s=1}^{n}\sum_{i=1}^{m}\int_0^{2\pi} f_{si}\,[\varphi(t + \alpha_1, c), \ldots, \varphi(t + \alpha_m, c);\ t, 0] \times$$
$$\times\, \psi_s(t + \alpha_i, c)\,\delta_{il}\,dt = 0 \quad (l = 1, \ldots, m)$$

or, finally,

$$P_l(\alpha_1, \ldots, \alpha_m) =$$
$$= \sum_{s=1}^{n}\int_0^{2\pi} f_{sl}\,[\varphi(t + \alpha_1, c), \ldots, \varphi(t + \alpha_m, c);\ t, 0]\,\psi_s(t + \alpha_l, c)\,dt = 0$$
$$(l = 1, \ldots, m) \quad (1.15)$$

If under the condition that the roots of Equation (1.14) are not multiple, system (1.15) admits of a solution for which the inequality

$$\frac{\partial(P_1, \ldots, P_m)}{\partial(\alpha_1, \ldots, \alpha_m)} \neq 0 \quad (1.16)$$

is satisfied, then for sufficiently small μ to this solution there corresponds a synchronous motion of the interconnected system with a unique frequency, the sequential approximations to which can be sought in the form of a formal series in powers of μ

$$x_{si}(t) = \varphi_s(t + \alpha_i, c) + \mu x_{si}^{(1)} + \mu^2 \ldots \quad (1.17)$$

2. Passing on to the study of the stability of solution (1.6), we set up the variational equation of the complete system (1.1),

$$\frac{dz_{si}}{dt} = \sum_{r=1}^{n} p_{sr}(t + \alpha_i) z_{ri} + \mu \sum_{r=1}^{n} \sum_{j=1}^{m} q_{sr}^{(ij)} z_{rj} + \mu^2 \ldots \quad \begin{pmatrix} s = 1, \ldots, n \\ i = 0, \ldots, m \end{pmatrix} \quad (2.1)$$

where

$$q_{sr}^{(ij)} = \sum_{\alpha=1}^{n} \left(\frac{\partial^2 X_s(x_i)}{\partial x_{ri} \partial x_{\alpha j}} \right) x_{\alpha j}^{(1)} + \left(\frac{\partial f_{si}(x_1, \ldots, x_m, t, 0)}{\partial x_{rj}} \right) \quad (2.2)$$

The parantheses in (2.2) signify that the corresponding quantities are calculated for the generating approximation. By using the substitution of N.A. Artem'ev [4], we shall seek particular solutions of system (2.1) in the form

$$z_{si} = e^{a(\mu)t} \eta_{si} \quad (2.3)$$

where η_{si} is the 2π-periodic solution of the system

$$\frac{d\eta_{si}}{dt} = \sum_{r=1}^{n} p_{sr}(t + \alpha_i) \eta_{ri} + \mu \sum_{r=1}^{n} \sum_{j=1}^{m} q_{sr}^{(ij)} \eta_{rj} - a(\mu) \eta_{si} + \mu^2 \ldots \quad (2.4)$$

Let us confine ourselves to the consideration of the spproximations to the critical roots only and assume $a(0) = 0$. Then when $\mu = 0$, system (2.4) turns ito the variational equation system of the generating system (1.4) which admits of a group of $2m$ solutions (1.6) and (1.7) corresponding to the critical $2m$-fold zero root. Thus, all the indices of the elementary divisors of the characteristic determinant of the generating system which correspond to the zero root, are equal two. Consequently [4], the characteristic indices of system (2.1), which vanish when $\mu = 0$, must necessarily be sought in the form of a series in powers of $\mu^{\frac{1}{2}}$

$$a(\mu) = a\mu^{1/2} + a_2\mu + a_3\mu^{3/2} + \ldots \quad (2.5)$$

If the 2π-periodic solution of system (2.4) is sought in the form of the series

$$\eta_{si} = \eta_{si}^{(0)} + \mu^{1/2}\eta_{si}^{(1)} + \mu\eta_{si}^{(2)} + \mu^{3/2} \ldots \quad (2.6)$$

then the periodic zero approximation has the form

$$\eta_{si}^{(0)} = M_i \varphi_s(t + \alpha_i, c) \qquad (M_i = \text{const}) \quad (2.7)$$

The system of equations for determining the first approximation

$$\frac{d\eta_{si}^{(1)}}{dt} = \sum_{r=1}^{n} p_{sr}(t + \alpha_i) \eta_{ri}^{(1)} - a_1 M_i \varphi_s(t + \alpha_i, c) \quad (2.8)$$

according to (1.9), also admits of a 2π-periodic solution

$$\eta_{si}^{(1)} = -a_1 M_i \frac{2\pi}{dT(c)/dc} y_s(t + \alpha_i, c) + N_i \varphi_s^{\cdot}(t + \alpha_i, c) \quad (2.9)$$

depending on the $2m$ constants M_i and N_i.

For $\eta_{si}^{(2)}$ we get Equations

$$\frac{d\eta_{si}^{(2)}}{dt} = \sum_{r=1}^{n} p_{sr}(t+\alpha_i)\,\eta_{ri}^{(2)} + \sum_{r=1}^{n}\sum_{j=1}^{m} M_j q_{sr}^{(ij)} \varphi_r^{\cdot}(t+\alpha_j, c) + \tag{2.10}$$

$$+ a_1^2 M_i \frac{2\pi}{dT(c)/dc} y_s(t+\alpha_i, c) - a_2 M_i \varphi_s^{\cdot}(t+\alpha_i, c) - a_1 N_i \varphi_s^{\cdot}(t+\alpha_i, c)$$

Without proof let us write out the known identity [3]

$$\sum_{s,\,r=1}^{n} \int_0^{2\pi} q_{sr}^{(lj)} \varphi_r^{\cdot}(t+\alpha_j, c)\,\psi_s(t+\alpha_l, c)\,dt = \frac{\partial P_l}{\partial \alpha_j} \tag{2.11}$$

and in correspondence with (1.13) let us introduce the following notation

$$\sum_{s=1}^{n} y_s(t, c)\,\psi_s(t, c) = \sum_{s=1}^{n} \frac{\partial \varphi_s(0, c)}{\partial c} \psi_s(0, c) = k \tag{2.12}$$

Then, the conditions for the existence of a 2π-periodic solution of system (2.10) will take the form

$$\sum_{j=1}^{m} \left(\frac{\partial P_l}{\partial \alpha_j} + \varkappa \delta_{lj}\right) M_j = 0 \qquad (l = 1, \ldots, m) \quad \left(\varkappa = a_1^2 \frac{4\pi^2 k}{dT(c)/dc}\right) \tag{2.13}$$

The condition for the solution of system (2.13) to be nontrivial is

$$\Delta(\varkappa) = \begin{vmatrix} \partial P_1/\partial \alpha_1 - \varkappa \ldots \partial P_1/\partial \alpha_m \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \partial P_m/\partial \alpha_1 \quad \ldots \partial P_m/\partial \alpha_m - \varkappa \end{vmatrix} = 0 \tag{2.14}$$

Obviously, for the considered synchronous state to be stable it is necessary that all the roots of Equation (2.14) be real and satisfy condition

$$4\pi^2 a_1^2 = \frac{1}{R}\frac{dT(c)}{dc} \varkappa < 0 \tag{2.15}$$

The necessary and sufficient stability conditions may be obtained after computing the second approximations a_2 to the characteristic indices, and also the estimates of the characteristic indices, which, when $\mu = 0$, turn into the pure imaginary roots of the variational equations of the generating system.

However, without dwelling on these questions, let us briefly consider the case of internal synchronization under the assumption that the interconnected system of objects is self-contained in the whole. In this case the functions f_{sl} do not depend explicitly on time and relation (1.14) does not hold. The conditions for the existence of a synchronous state in the system

$$P_l(\alpha_1, \ldots, \alpha_m, c) = \qquad (l = 1, \ldots, m) \tag{2.16}$$

$$= \sum_{s=1}^{n} \int_0^{T(c)} f_{sl}\,[\varphi(t+\alpha_1, c), \ldots, \varphi(t+\alpha_m, c); 0]\,\psi_s(t+\alpha_l, c)\,dt = 0$$

satisfy the relations

$$P_l(\alpha_1, \ldots, \alpha_m, c) = P_l(\alpha_1 + \alpha, \ldots, \alpha_m + \alpha, c) \tag{2.17}$$

where, generally, α is an arbitrary constant. These conditions serve to determine uniquely the parameter c and the differences of the generating phases $\alpha_1 - \alpha m, \ldots, \alpha_{s-1} - \alpha_s$. As before, the necessary stability condition has the form (2.15). However, one of the roots of determinant (2.14) reduces to zero because of (2.17), but this does not influence the stability by virtue of the Andronov-Vitt theorem on the stability of periodic solutions of self-contained systems [5].

3. Let us consider a system of m nearly similar objects located on a supporting body of general shape which in this case will play the role of the coupling. The dynamic properties of the supporting body are taken as known. We shall assume that for the two points M and N of the supporting body we can uniquely determine a symmetric tensor of the second rank, $\mathbf{K}(M, N) = \mathbf{K}(N, M)$, such that the displacement $\mathbf{u}$ of the point M under the action of a force $\mathbf{Q}$ located at point N will be

$$\mathbf{u} = \mathbf{K}\,(M, N)\,\mathbf{Q} \tag{3.1}$$

Generally, the objects are almost conservative and are located at points M_i ($i = 1, \ldots, m$) of the supporting body. Here, neglecting the influences of object rotation inertia, we shall characterize the action of the ith object on the coupling uniquely by the inertial force $\mathbf{F}_i$

The equation in Fredholm form of the small oscillations of the coupling is

$$\begin{aligned} \mathbf{u}\,(M, t) = \int_{(V)} \mathbf{K}\,(M, N)\Big[\rho\,(N)\frac{\partial^2 \mathbf{u}\,(N, t)}{\partial t^2} + \mathbf{R} + \\ + \sum_{i=1}^{m} \mathbf{F}_i \delta\,(N, M_i) + \mathbf{f}\,(N, t)\Big]\,dV_N \end{aligned} \tag{3.2}$$

where $\mathbf{u}(M, t)$ is the displacement of the point M of the supporting body, $\rho(N)$ is the mass of a unit volume, $\mathbf{R}$ is the density of the dissipation force, $\mathbf{f}(N, t)$ is the density of the external action on the supporting body, which is 2π-periodic in time. The integration is carried out over the whole volume V of the supporting body, and $\delta(N, M)$ is the generalized delta-function satisfying condition

$$\int_{(\sigma)} \delta\,(N, M)\,dV_N = \begin{cases} 1 & \text{if} \quad M \in \sigma \\ 0 & \text{if} \quad M \,\overline{\in}\, \sigma \end{cases}$$

The motion of nearly similar objects in a moving coordinate system which is rigidly connected to some small neighborhood of the point of fastening, will be characterized by the relative generalized coordinates q_{si} ($s = 1, \ldots, n$). By definition, the kinetic energy of the ith object will be

$$\begin{aligned} T_i = \frac{1}{2}\sum_{\nu=1}^{N} m_{\nu i}\left[\frac{d\mathbf{r}_{\nu i}\,(q_i)}{dt} + \frac{\partial \mathbf{u}\,(M_i, t)}{\partial t}\right]^2 = \\ = T_{0i}\,(q_i, q_i^{\cdot}) + \frac{\partial \mathbf{u}\,(M_i, t)}{\partial t}\frac{d\mathbf{S}\,(q_i)}{dt} + \frac{1}{2} m_i \left(\frac{\partial \mathbf{u}\,(M_i, t)}{\partial t}\right)^2 \end{aligned} \tag{3.3}$$

where $\mathbf{r}_{\nu i}(q_i)$ is the radius vector of the mass element $m_{\nu i}$ of the object with number i in the moving coordinate system (3.4)

$$m_i = \sum_{\nu=1}^{N} m_{\nu i}, \quad \mathrm{S}_i(q_i) = \sum_{\nu=1}^{N} m_{\nu i}\, \mathbf{r}_{\nu i}(q_i), \quad T_{0i}(q_i, q_i^{\cdot}) = \frac{1}{2} \sum_{s,r=1}^{n} A_{sr}^{(i)}(q_i)\, q_{si}^{\cdot} q_{ri}^{\cdot}$$

Obviously, m_i is the total mass of the ith object, $\mathbf{S}_i(q_i)$ is the vectorial static moment of the object relative to the point of fastening, and $T_{0i}(q_i, q_i^{\cdot})$ is the relative kinetic energy, being a uniform, positive-definite quadratic form in the generalized velocities $q_{si}^{\cdot}$. By assuming that the potential energy of the object depends only on the relative generalized coordinates,

$$\Pi_i = \Pi_i(q_i) \tag{3.5}$$

we get an expression for the force with which the object acts on the supporting body

$$F = \frac{d}{dt} \frac{\partial T_i}{\partial (\partial u^{\cdot}(M_i, t)/\partial t)} = \frac{d^2 \mathrm{S}_i(q_i)}{dt^2} + m_i \frac{\partial^2 \mathbf{u}(M_i, t)}{\partial t^2} \tag{3.6}$$

Equation (3.2) of small oscillations of the supporting body is rewritten as

$$\mathbf{u}(M, t) = \sum_{i=1}^{m} \mathbf{K}(M, M_i)\, \mathrm{S}_i^{\cdot\cdot}(a_i) + \tag{3.7}$$

$$+ \int_{(V)} \mathbf{K}(M, N) \cdot \left\{ \mathbf{R} + \mathbf{f}(N, t) + [\rho(N) + \sum_{i=1}^{m} \delta(N, M_i)\; m_i] \frac{\partial^2 \mathbf{u}(N, t)}{\partial t^2} \right\} dV_N$$

Furthermore, we shall assume that the influence tensor ot the supporting body can be represented as a bilinear expansion

$$\mathbf{K}(M, N) = -\sum_{j=1}^{\infty} \frac{\boldsymbol{\theta}_j(M)\, \boldsymbol{\theta}_j(N)}{\lambda_j} \tag{3.8}$$

where λ_j are the eigenvalues and $\boldsymbol{\theta}_j(M)$ are the vectorial eigenfunctions of the body, satisfying the orthogonality and normality conditions

$$\int_{(V)} [\rho(N) + \sum_{i=1}^{m} m_i \delta(N, M_i)]\, \boldsymbol{\theta}_j(N) \cdot \boldsymbol{\theta}_l(N)\, dV_N = \delta_{jl} \quad (j, l = 1, 2, \ldots, \infty) \tag{3.9}$$

We shall seek a solution of Equation (3.7) in the form of a series

$$\mathbf{u}(M, t) = \sum_{j=1}^{\infty} \boldsymbol{\theta}_j(M)\, u_j(t) \tag{3.10}$$

Let us introduce the following assumption on the nature of the force damping the oscillations of the supporting body

$$\int_{(V)} \boldsymbol{\theta}_j(N) \cdot \mathbf{R}\, dV_N = \beta \lambda_j u_j^{\cdot} \tag{3.11}$$

where β is a positive constant. In a specific sense this hypothesis assumes the proportioality of the internal resistance force to the rate of change of the resisting force, and in the particular case of the ordinary girder, degenerates to the well-known Voigt hypothesis. For a purely qualitative estimate of the influence of the resistence force, such an assumption is completely acceptable; all the more so since the coefficient β, in what follows, will be assumed to be a quantity of the order of the small coupling parameter.

After substituting series (3.10) into Equation (3.7) and transforming with due regard to (3.8), (3.9) and (3.11), we arrive at the infinite system

$$u_j^{\cdot\cdot} + \beta\lambda_j u_j + \lambda_j u_j + \sum_{i=1}^{m} \theta_j (M_j)\cdot S_i^{\cdot\cdot}(q_i) + f_j(t) = 0 \quad (j = 1, 2, \ldots) \tag{3.12}$$

Here

$$f_j(t) = \int_{(V)} \theta_j(N)\, \mathbf{f}(N, t) dV_N \tag{3.13}$$

The Lagrange equations for the motions of the objects, set up by the usual methods with due regard to the relations

$$\frac{\partial}{\partial q_{si}^{\cdot}} \frac{dS_i(q_i)}{dt} = \frac{\partial S_i(q_i)}{\partial q_{si}}, \qquad \frac{d}{\partial t} \frac{\partial S_i(q_i)}{dt_{si}} = \frac{\partial}{\partial q_{si}} \frac{dS_i(q_i)}{dt} \tag{3.14}$$

have the form

$$\frac{d}{dt} \frac{\partial T_{0i}(q_i, q_i^{\cdot})}{\partial q_{si}} - \frac{\partial T_{0i}(q_i, q_i^{\cdot})}{\partial q_{si}} + \frac{\partial \Pi_i(q_i)}{\partial q_{si}} +$$

$$+ \sum_{j=1}^{\infty} \theta_j(M_i) \frac{\partial S_i(q_i)}{\partial q} u_i^{\cdot\cdot} = Q_{si}(q_i, q_i^{\cdot}) \quad \begin{pmatrix} i = 1, \ldots, m \\ s = 1, \ldots, n \end{pmatrix} \tag{3.15}$$

and, the generalized forces Q_{si}, characterizing the inflow and loss of energy, by virtue of the original assumption that the objects are nearly conservative, are assumed to be quantities of the order of the small coupling parameter.

Let us introduce the new canonic variables q_{si} and

$$p_{si} = \partial T_{0i}(q_i, q_i^{\cdot}) / \partial q_{si}^{\cdot}$$

and the Hamiltonian function of the object $H_i(q_i, p_i)$. Here, by virtue of the fact that the objects are nearly similar

$$H_i(q_i, p_i) = H(q_i, p_i) + \Delta H(q_i, p_i), \quad S_i(q_i) = S(q_i) + \Delta S_i(q_i) \tag{3.16}$$

$$Q_{si}(q_i, q_i^{\cdot}) = Q_s(q_i, q_i^{\cdot}) + \Delta Q_{si}(q_i, q_i^{\cdot})$$

In the latter relations the second terms are assumed to be quantities of the order of the small coupling parameter relative to the first. Finally, the equations of motion of the interconnected system of objects, with an accuracy up to a quantity of the first order of smallness inclusively, have the form

$$q_{si}^{\cdot} = \frac{\partial H(q_i, p_i)}{\partial p_{si}} + \left(\frac{\partial \Delta H_i(q_i, p_i)}{\partial p_{si}} \right) \qquad \begin{pmatrix} i = 1, \ldots, m \\ s = 1, \ldots, n \end{pmatrix} \tag{3.17}$$

(3.17) cont.

$$p_{si}^{\cdot} = -\frac{\partial H(q_i, p_i)}{\partial q_{si}} + \left(-\frac{\partial \Delta H_i(q_i, p_i)}{\partial q_{si}} + Q_s(q_i, q_i^{\cdot}) - \sum_{j=1}^{\infty} (\theta_i(M_i)\frac{\partial S(q_i)}{\partial q_{si}} u_j^{\cdot\cdot}\right)$$

$$u_j^{\cdot\cdot} + \lambda_j u_j = -\sum_{i=1}^{m} (\theta_j(M_i) S^{\cdot\cdot}(q_i) - f_j(t) - (\beta\lambda_s u_s^{\cdot} + \sum_{i=1}^{m} \theta_j(M_i)\Delta S^{\cdot\cdot}{}_i(q_i))$$

$$(j = 1, 2, \ldots)$$

In (3.17) the terms in parantheses are assumed to be small and, moreover, the group of conservative terms, reflecting the action of the coupling on the object, is small by virtue of the assumption that the coupling is weak.

4. In the generating approximation we have m uncoupled similar self-contained conservative subsystems

$$(q_{si}^{\circ})^{\cdot} = \frac{\partial H(q_i^{\circ}, p_i^{\circ})}{\partial p_{si}^{\circ}}, \qquad (p_{si}^{\circ})^{\cdot} = -\frac{\partial H(q_i^{\circ}, p_i^{\circ})}{\partial q_{si}^{\circ}} \qquad (i = 1, \ldots, m) \quad (4.1)$$

admitting of $T(c)$-periodic solutions

$$q_{si}^{\circ} = q_s(t + \alpha_i, c) \qquad p_{si}^{\circ} = p_s(t + \alpha_i, c) \quad (4.2)$$

Each of the subsystems (4.1) admit of an energy integral which for the considered solution (4.2) has the form

$$H(q, p) = h(c) \quad (4.3)$$

Here the energy constant h is positive and is an analytic function of parameter c (see [3]). The generating 2π-periodic solution of (4.1) is characterized by the relation

$$T(c) = 2\pi \quad (4.4)$$

If the functions $v_j(t)$ and $w_j(t)$ are 2π-periodic solutions of Equation

$$v_j^{\cdot\cdot} + \lambda_j v_j = f_j(t), \qquad w_j^{\cdot\cdot} + \lambda_j w_j = S^{\cdot\cdot}(q) \qquad (j = 1, 2, \ldots) \quad (4.5)$$

then for the oscillations of the supporting body in the generating approximation we have

$$u_j^{\circ} = -v_j(t) - \sum_{i=1}^{m} \theta_j(M_i) \cdot w_j(t + \alpha_i) \quad (4.6)$$

It is clear that we should assume the existence of the inequality $\lambda_i \neq k$ (k is an integer) and by the same token we should exclude from consideration the case of resonance in any of the normal coordinates of the supporting body. The variational equations of the generating system have the m periodic solutions

$$q_s^{\cdot}(t + \alpha_i, C)\,\delta_{ik}, \qquad p_s^{\cdot}(t + \alpha_i, c)\,\delta_{ik} \qquad (k = 1, \ldots, m)$$

with periods of 2π. The corresponding family of 2π-periodic solutions of the conjugate system will be

$$-p_s^{\cdot}(t + \alpha_i, c)\,\delta_{ik}, \qquad q_s^{\cdot}(t + \alpha_i, c)\,\delta_{ik}$$

According to (1.15) the system of equations for determining the generating phases $\alpha_1, \ldots, \alpha_n$, after some manipulations with due regard to the fact that

$$\int_0^{2\pi} \sum_{s=1}^{n} \left\{ \frac{\partial \Delta H_i(q, p)}{\partial p_s} p_s^{\cdot} + \frac{\partial \Delta H_i(q, p)}{\partial q_s} q_s^{\cdot} \right\} dt = 0 \tag{4.7}$$

acquires the form

$$P_k(\alpha_1, \ldots, \alpha_m, c) = Q(c) + \Phi_k(\alpha_k, c) + R_k(\alpha_1, \ldots, \alpha_m, c) = 0 \quad (k = 1, \ldots, m) \tag{4.8}$$

where

$$Q(c) = \int_0^{2\pi} \sum_{s=1}^{n} Q_s(q, q^{\cdot})\, q_s^{\cdot}\, dt$$

$$\Phi_k(\alpha_k, c) = \int_0^{2\pi} \sum_{j=1}^{\infty} S^{\cdot}[q(t + \alpha_k, c)] \cdot \theta_j(M_k)\, v_j(t)\, dt \tag{4.9}$$

$$R_k(\alpha_1, \ldots, \alpha_m, c) = \int_0^{2\pi} \sum_{j=1}^{\infty} \sum_{r=1}^{m} S^{\cdot}[q(t + \alpha_k, c)] \cdot \theta_j(M_k)\, \theta_j(M_r) \cdot w_j^{\cdot\cdot}(t + \alpha_r)\, dt$$

In the case of internal synchronization, the density of the external action on the coupling, $f(M, t) = 0$ and, as has already been said above, Equation (4.4) does not hold. Here the conditions for the existence of a synchronous state in the system

$$P_k(\alpha_1, \ldots, \alpha_m, c) = Q(c) + R_k(\alpha_1, \ldots, \alpha_m, c) = 0 \quad (k = 1, \ldots, m) \tag{4.10}$$

because of (2.18) must be considered as a system of equations in the unknowns $c, \alpha_1 - \alpha_m, \ldots, \alpha_{m-1} - \alpha_m$. The functions Q and R_k in (4.10) are determined from (4.9); however, here the integration is between the limits 0 and $T(c)$. Turning to the investigation of the quantity R_k, we integrate the last relation in (4.9) by parts, after which, taking (4.5) into account, we get

$$R_k(\alpha_1, \ldots, \alpha_m, c) = -\int_0^{2T(c)} - \sum_{j=1}^{\infty} \sum_{r=1}^{m} [w_j^{\cdot\cdot}(t + \alpha_k) + \lambda_j w_j^{\cdot}(t + \alpha_k)] \cdot \theta_j(M_k) \times \theta_j(M_r) \cdot w_j^{\cdot}(t + \alpha_r)\, dt \tag{4.11}$$

From (4.11) it immediately follows that

$$\sum_{k=1}^{m} R_k(\alpha_1, \ldots, \alpha_m, c) = -\int_0^{T(c)} \sum_{j=1}^{\infty} (u_j^{\cdot\cdot\circ} + \lambda_j u_j^{\circ})\, u_j^{\cdot\circ} dt = 0 \tag{4.12}$$

If now we sum (4.10) with respect to k, then because of (4.12) we shall have

$$Q(c) = 0 \tag{4.13}$$

The frequency of the synchronous state, when couplings appear in the generating approximation, is obviously not shifted. The equations for the determination of the generating phases

$$R_k(\alpha_1, \ldots, \alpha_m, c) = 0 \quad (k = 1, \ldots, m) \tag{4.14}$$

because of (4.11) have the particular solution $\alpha_1 = \ldots = \alpha_s = \alpha$ if

$$\theta_j(M_k) = \theta_j(M_r)\, b_{kr}^{(j)}$$

where $b_{kr}^{(j)}$ are some scalar multipliers.

By setting up the expression for the mean from the period Lagrange function of the supporting body in the generating approximation

$$\Lambda(\alpha_1, \ldots, \alpha_m, c) = \int_0^{T(c)} \frac{1}{2} \sum_{j=1}^{\infty} (u_j^{\cdot \circ 2} - \lambda_j u_j^{\circ 2})\, dt \qquad (4.15)$$

it is easy to verify the equality

$$R_k = -\frac{\partial \Lambda}{\partial \alpha_k} \qquad (k = 1, \ldots, m) \qquad (4.16)$$

Turning to a stability investigation, let us note that the constant k has a distinct physical meaning for conservative objects.

Indeed, of the basis of (2.12), (4.1) and (4.3),

$$k = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_s}\bigg|_{t=0} \frac{\partial q_s(0, c)}{\partial c} + \frac{\partial H}{\partial p_s}\bigg|_{t=0} \frac{\partial p_s(0, c)}{\partial c} \right) = \frac{dh(c)}{dc} \qquad (4.17)$$

Thus, for the stability of the obtained synchronous state in the interconnected self-contained system, it is necessary that all the roots of Equation

$$\begin{vmatrix} \dfrac{\partial^2 \Lambda}{\partial \alpha_1^2} - \varkappa & \ldots & \dfrac{\partial^2 \Lambda}{\partial \alpha_1 \partial \alpha_m} \\ \ldots & \ldots & \ldots \\ \dfrac{\partial^2 \Lambda}{\partial \alpha_1 \partial \alpha_m} & \ldots & \dfrac{\partial^2 \Lambda}{\partial \alpha_m^2} - \varkappa \end{vmatrix} = 0 \qquad (4.18)$$

except one which equals zero, should satisfy condition

$$\frac{dT(c)/dc}{dh(c)/dc}\, \varkappa < 0 \qquad (4.19)$$

The reality of the roots of determinant (4.18) is ensured by its symmetry.

In the case of mechanical vibrators [1] the gyration period decreases with a growth in the vibrator energy. Consequently $dT/dh < 0$ and we arrive at the minimum condition first formulated by Blekhman and Lavrov [6] and then proved by Blekhman [7] conformably to systems of mechanical vibrators or their mathematical analogs. This condition incidentally is not only necessary but also sufficient in the mentioned problem because of the specific method of introducing the small parameter (which was different from the method used in the present paper).

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